Mathematics in Grade 4 Lesson Learned 1
Translating Between and Among Representations

Students were challenged when translating and moving flexibly between and among all representations of a concept. Students need to be encouraged to translate between words, pictures and symbols. If asked to represent a question using representations, (words, symbols and/or pictures), students usually only provide symbols. They do not seem to realize that they can use varied representations (words, symbols and/or pictures) when solving a question.

Mathematics should be viewed as an integrated whole rather than as the study of separate strands or units. Connections must also be made between and among the different representational modes – contextual, concrete, pictorial, linguistic/verbal, and symbolic. The process of making connections, in turn, facilitates a child's learning.

"Children who have difficultly translating a concept from one representation to another are the same children who have difficulty solving problems and understanding computations. Strengthening the ability to move between and among these representations improves the growth of children's concepts" (Van De Walle, John A. 2001, *Elementary and Middle School Mathematics*, Fourth Edition, p. 34).

One way to encourage children to use multiple representations is to explicitly ask for them.
Ask questions such as

- How many ways can you show the number 20 using words, pictures, models, and numbers?
- How many ways can you write 75?
- Can you represent a rectangle as a combination of other shapes?
- Can you represent this line plot as a bar graph?
- Can you use an equation to represent how you thought about this story problem?

A. What conclusions can be drawn from the Nova Scotia Assessment: Mathematics in Grade 4?

Historically, our students were challenged when asked to translate between and among representations of a concept. However, the student results of the 2016–2017 Nova Scotia Assessment Mathematics in Grade 4 reveals that there has been improvement over time. Students now have translating between and among representation well under control. We found that students have a good understanding of basic facts and procedures, but when given application items, they appear to want to rush to symbolic. For example, when problem-solving, students are able to understand the context of the question but many are not able to translate between the representations (translating from words to pictures or symbolic to pictures, etc.).

B. Do students have any misconceptions or errors in their thinking?

If asked to represent a question using words, symbols and/or pictures, students usually only provided symbols. They do not seem to realize that they can use various representations (words, symbols and/or pictures) when solving a question. For example, when given an analysis question concerning the perimeter (distance around) a named shape, students struggle with knowing how many sides the named shape would have, unless the actual shape is shown in the question as a picture/diagram. Students should be encouraged to translate between the name of the shape (words) and a picture. Rather than relying on the question itself, students could draw the picture and check how many sides the named shape has.

The student results of the 2016–2017 Nova Scotia Assessment Mathematics in Grade 4 reveals that the students have translating between and among representation well under control.

C. What are the next steps in instruction for the class and for individual students?

Next steps in instruction should provide opportunities for students to use representations to communicate mathematical ideas. They should have experiences selecting, applying, modeling, and translating among mathematical representations to solve problems. The five representations of a concept are contextual, concrete, pictorial, symbolic, and verbal (written/oral). All five representations should remain a focus for students.

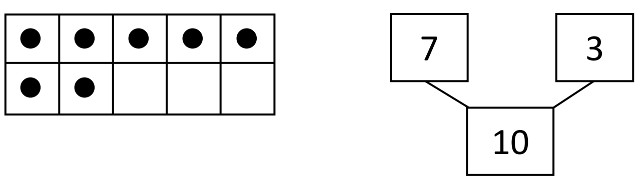
"Representational competence (Novick, 2004) for students is knowing how and when to use particular mathematical representations. A key aspect of understanding mathematics means not only knowing how to use a representation during problem-solving situations but also being able to make connections between representations" (*Teaching Children Mathematics*. NCTM, August 2010, p. 40). When students are able to move between and among the different representations of a concept, we say that they have concept attainment.

There are three specific strategies that provide opportunities to support students' development of representational competency

- "Engaging in dialogue about the explicit connections between representations
- Alternating directionality of the connections made among representations
- Encouraging purposeful selection of representation"
  (*Teaching Children Mathematics*. NCTM, August 2010, p. 40)

For example, during instruction students focus on specific parts of one representation and think about the correspondence with parts in another. Teachers should ask questions that require students to translate between parts. Alternating directionality supports students' thinking with various representations.
For example, consider the activity below, All About Ten (*Teaching Children Mathematics*. NCTM, August 2010, p. 44).

All About Ten: Fill in the box diagram to show the two parts of ten shown in the ten-frame. Write number sentences to match. [Below is the completed activity.]



7 + 3 = 10	10 = 7 + 3
3 + 7 = 10	10 = 3 + 7
10 – 7 = 3	3 = 10 – 7
10 – 3 = 7	7 = 10 – 3

Instructional Strategies

"Three specific instructional strategies create opportunities that may support students' development of representational competence …" (*Teaching Children Mathematics*. NCTM, August 2010, p. 40).

1. Students need to **engage** in dialogue about the explicit connections between representations. Pose questions to students such as
   - How is the number 10 represented in each diagram?
   - How are the three representations of the number the same? How are they different?
   - Can you show the number 10 in a different way?

2. Students need to be encouraged to **alternate directionality** in order to make connections among representations. When discussing multiple representations, ask focused questions such as
   - Can you describe the 3 in the ten-frame, in the box diagram, and in each of the equations?
   - What meaning does the 3 have in each of the diagrams?

   "The **directionality** of the connections made between the representations and the problem situation is another important feature of representational competence. For example, translating from a box diagram to a ten-frame, and vice versa, promotes the use of different mathematical thought processes**.** An important aspect of developing understanding of mathematics means not only knowing how to use a representation during problem solving situations but also being able to move flexibly between different representations, making connections from one representation to the other, and vice versa."
   (*Teaching Children Mathematics*. NCTM, August 2010, p. 44)

3. Students need to be **encouraged** to purposefully select the most appropriate representation. "Encourage students to consider the suitability of a representation. Discuss a variety of reasons to use particular representations, including but not limited to the following: efficiency, accuracy, ease of use, appropriateness with respect to the problem context, and student preference. By comparing the use of multiple representations for the same problem, students can more easily see the suitablility of one representation over another" (*Teaching Children Mathematics*. NCTM, August 2010, p. 46).

D. What are the most appropriate methods and activities for assessing student learning?

The strands Measurement (M) and Geometry (G) will be used to represent some of the appropriate methods and activities for assessing student learning.

Encourage students to share their thinking, their strategies, and their solutions. Rich discussion fosters the development of alternative problem-solving strategies.

Examples:

1. Peter found a stick that was more than a 1 m long.
How many centimetres long could the stick be?

- O 80 cm
- O 90 cm
- O 100 cm
- O 115 cm

Show how you know using words, pictures, and/or symbols.

2. The perimeter (distance around) of a regular pentagon is 20 cm.
How long is each side?

- O 4 cm
- O 5 cm
- O 20 cm
- O 25 cm

Show how you know using words, pictures, and/or symbols.

3. I am a 3-D object.
I have 5 faces.
I have 5 vertices.
I have 8 edges.
Which shape am I?

- O cube
- O sphere
- O square-based pyramid
- O triangular-based pyramid

Show how you know using words, pictures, and/or symbols.

Mathematics in Grade 4 Lesson Learned 2
Representing and Partitioning Whole Numbers

Students were challenged when asked to apply their knowledge of basic facts and skills and to represent a situation or the steps in a procedure when given application questions. They struggled with translating between representations of a concept (contextually, concretely, pictorially, symbolically, and verbally). When students are working to partition whole numbers and to perform operations, it is very important for students to understand that numbers can be broken down into two or more parts in many different ways.

A. What conclusions can be drawn from the Nova Scotia Assessment: Mathematics in Grade 4?

We noticed that students were able to do well when explicitly given all the information needed to do the question. Students performed well on knowledge questions that required them to use basic skills, symbolic procedures, and factual knowledge. For example, students were successful when asked to calculate 61 + 35. Students were successful problem solvers and performed well on questions that required analysis and non-routine problem-solving.

However, our assessment information also shows that many students experienced challenges with application questions. Our students were challenged when asked to apply their knowledge of basic facts and skills to a context. They also struggled with translating between representations of a concept (contextually, concretely, pictorially, symbolically, and verbally).

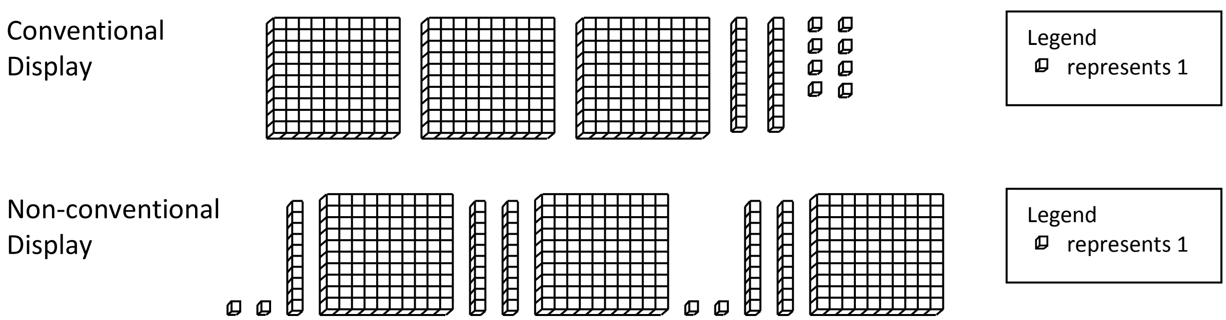
B. Do students have any misconceptions or errors in their thinking?

Students were very capable of performing well with partitioning numbers when given knowledge questions. For example, students represented a number using base-ten blocks in a conventional display. However, the difficulty appeared when students were asked to apply their knowledge or to represent a situation or the steps in a procedure when given application questions. For example, students could correctly represent a number, such as 75 using a conventional display of base-ten blocks but were challenged when asked to partition a number in a variety of ways, such as 70 + 5, 50 + 25, or 60 + 12 + 3, 25 + 25 + 20. Many students have the misconception that these are expressions that have an answer of 75, and do not understand that these also represent four ways of writing 75. An expression names a number. Sometimes an expression is a number such as 150. Sometimes an expression shows an arithmetic expression, such as 125 + 25. 150 may also be represented by its partitions, such as 80 + 70, 100 + 50, and 50 + 50 + 50. Numbers can also be represented by a difference expression, such 175 – 25.

The student result of the 2016–2017 Nova Scotia Assessment Mathematics in Grade 4 reveals that the students have representing and partitioning of numbers well under control.

C. What are the next steps in instruction for the class and for individual students?

When students translate between and among the five representations of a concept (contextual, concrete, pictorial, symbolic and verbal), we say that they have concept attainment. Students need numerous experiences representing numbers to 1000 and translating between and among these representations of a concept to strengthen their knowledge. They need many experiences with base-ten materials, pictures such as number lines and tallies, ten-frames, words, and contexts to conceptualize a number being made up of two or more parts. It is extremely important that students have opportunities to view and create numbers using conventional and non-conventional displays of base-ten blocks.



Partitioning numbers using the models above supports students' ability to recognize that any number can be partitioned into two or more parts. It also helps students develop part-part-whole thinking. Although it is important for students to experience a variety of partitions including traditional expanded notation (425 = 400 + 20 + 5), they should also continue to experience partitions such as 424 + 1, 325 + 100, 200 + 200 + 10 + 10 + 5. This is the most important understanding that can be developed about number relationships.

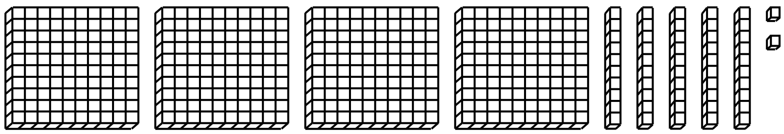
D. What are the most appropriate methods and activities for assessing student learning?

Below are some sample questions related to representing and partitioning whole numbers which will be used to represent some of the appropriate methods and activities for assessing student learning.

Encourage students to share their thinking, their strategies, and their solutions. Rich discussion fosters the development of alternative problem-solving strategies.

Examples:

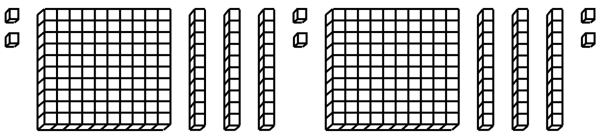
1. What number do these base-ten blocks represent?



Legend
represents 1

Write the number, _________.

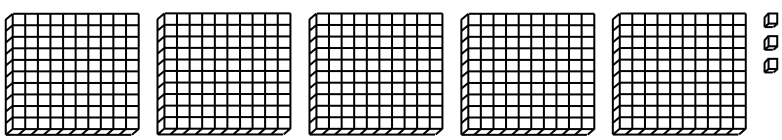
2. What number do these base-ten blocks represent?



Legend
represents 1

Write the number, __________.

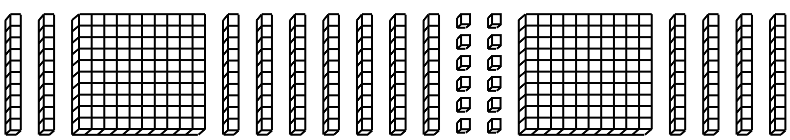
3. What number do these base-ten blocks represent?



Legend
represents 1

Write the number, _________.

4. What number do these base-ten blocks represent?



Legend
represents 1

Write the number, _________.

5. Draw a picture of base-ten blocks to show 236 in 3 different ways.

Legend
□ represents 1

6. The number 358 is the same as:

○ 100 + 100 + 50 + 8 + 100
○ 300 + 5 + 8
○ 400 – 58
○ 3 + 5 + 8

7. Choose the number that is equal to thirty-one tens.

○ 31
○ 301
○ 310
○ 3010

8. The number 642 is the same as:

○ 5 hundreds, 2 tens, and 14 ones
○ 64 tens and 2 ones
○ 6 tens and 42 ones
○ 6 hundreds, 20 tens and 4 ones

9. Write three expressions that can be used to represent 53.

53 is the same as ____

53 is the same as ____

53 is the same as ____

10. Write the following number in words:

263 ____

373 ____

487 ____

597 ____

Mathematics in Grade 4 Lesson Learned 3
Whole Number Operations

Students were challenged when asked to apply basic skills, knowledge, and computational procedures to application and analysis questions. Students need to be able to apply the higher order thinking skills of problem solving, creativity, and reasoning to do application and analysis items. Students should have experiences with all of the story structures for addition and subtraction. Students also need to be encouraged to estimate before calculating an answer to a question.

The Mathematics curriculum documents for grades Primary to 3 note that a true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Number sense develops when students connect numbers to real-life experiences and use benchmarks and referents. This results in students who are computationally fluent, flexible with numbers, and have intuition about numbers. The evolution of number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections. Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, and concepts.

A. What conclusions can be drawn from the Nova Scotia Assessment: Mathematics in Grade 4?

We noticed that students were able to do well when explicitly given all the information needed to do the question. Students performed well on knowledge questions that required them to use basic facts and skills, symbolic procedures, and factual knowledge. For example, students were able to solve 487 – 37 when the problem was presented symbolically.

However, when students were asked to apply basic skills, knowledge, and computational procedures to application and analysis questions, they were challenged. This appeared to be a theme throughout the assessment data. At times, students were not sure whether they should add, subtract, multiply, or divide when questions were presented in the context of a story problem. The assessment analysis also showed that our students did not understand the relationship between addition and subtraction. Many of the students were doing addition and subtraction questions as procedures and were not making any connection between these two operations.

B. Do students have any misconceptions or errors in their thinking?

Addition and Subtraction

When solving computation questions, many students performed better when solving the computation using an alternative algorithm, such as presenting the digits to be added or subtracted horizontally. Students did less well, when the traditional algorithm was used. This may result because the traditional algorithm focuses on single digits within the computation rather than thinking about the number as a whole.

Examples:

$67 - 32 = 35$

$$\begin{array}{r} 650 \\ \underline{-\ 421} \\ 229 \end{array}$$

Many students have the misconception that they always subtract the smaller number from the larger number. They apply this thinking regardless of whether the position of the number is in the subtrahend (a number which is to be subtracted from another number) or the minuend (a number from which another number is to be subtracted) in the question. Below are two examples.

451	(minuend)	5**0**9	(minuend)
– 231	(subtrahend)	– 3**8**9	(subtrahend)
220	(difference)	2**8**0	(difference)

In the first question, when subtracting the tens, 5 tens – 3 tens = 2 tens, (50 – 30 = 20) the student completes the question correctly using his/her understanding of subtracting the smaller number from the larger number. But in the second question, the student tries to use the same method (smaller number 0 subtracted from larger number 8) and they get an answer of 8 tens, which is incorrect.

At times, students forgot to regroup when adding. They often wrote a two-digit number where there should have only been one digit. Below is an example.

14**5**
+ 24**7**
38**12**

Some students misaligned the digits when recording their calculations and computed incorrectly. Ways to address these errors or misconceptions is to focus on developing personal strategies and alternative algorithms which tend to focus on the meaning of the number, rather than on individual digits.

Multiplication

It is important when developing multiplication and division concepts that children are able to explain the connection between their models and the story problems using verbal expressions such as “groups of,” “rows of” and “jumps of”.

Early in the development of the division concept, students should encounter situations involving remainders. Also, the connection between words and symbols must be carefully developed for division. For example, when writing 12 ÷ 4, we should say aloud, “How many groups of 4 are in 12?” As well, students should be familiar with both forms of the symbolic expression, and below are two examples.

$4\,\overline{)12}$ $\qquad$ $12 \div 4$

A misconception that students have about multiplication is that the product of two numbers is always greater than the sum of those two numbers. So, when they encountered expressions like 8 x 1, 8 x 0, and 2 x 2 where this did not apply, they were puzzled.

The student result of the **2016–2017 Nova Scotia Assessment Mathematics in Grade 4** shows the following data:

- 52% of the students were not able to round a 3-digit number to the nearest hundred.
- 53% of the students were not able to identify an array represented by a given number of rows and a given number of columns.
- 55% of the students were not able to round 2-digit and 3-digit numbers to the nearest hundred in a context.
- 56% of the students were not able to identify a multiplication fact represented by an array.
- 61% of the students had difficulty recognizing the place value of a number and being able to rename the same number in a variety of ways.
- 61% of the students were not able to subtract two 3-digit numbers correctly when the largest number in the subtraction question contained a zero in the tens place.

C. What are the next steps in instruction for the class and for individual students?

Students were challenged when applying estimation strategies. They knew how to round numbers in isolation but could not estimate sums and differences in context. One of the first steps in instruction is making sure that children are exposed to and understand how to estimate sums, differences, products, and quotients. Estimating allows students to predict answers, check their calculations, and ask themselves if their actual answer is reasonable. The factors that may influence estimating sums, differences, products, and quotients is the context and the numbers and operations involved.

Addition and Subtraction

Students need to learn that addition and subtraction are related. They are inverse operations; they undo each other. The basic facts of addition and subtraction do not have to be learned as separate facts. Below are some examples.

6 + 5 = 11, so 5 + 6 = 11	11 – 5 = 6, so 11 – 6 = 5
11 = 6 + 5, so 11 = 5 + 6	6 = 11 – 5, so 5 = 11 – 6

Addition and subtraction problems can be categorized based on the kinds of relationships they represent. It is important that all the story problem structures are presented and developed from students' experiences. Initial work with the story structures will focus on join and separate types of problems because students associate the actions in these problems with the operations. However, they must also experience addition and subtraction in part-part-whole and comparison situations. Please refer to the grade level appropriate curriculum documents (*Mathematics 1*, p. 64; *Mathematics 2*, p. 68; and *Mathematics 3*, p. 71) for more information about the story structures and instructional strategies.

Students will be expected to use and describe strategies to determine sums and differences using manipulatives and visual aids. Initial strategies include

- counting on or counting backwards
- one more or one less
- making ten
- doubles
- near doubles

Other strategies are described in the curriculum documents, *Mathematics 1*, *Mathematics 2*, and *Mathematics 3*.

Manipulatives can and should be used to model not only the above strategies but also model the story structures. Examples of manipulatives that can be used for this purpose include

- two-sided counters
- linking cubes
- game materials (number cubes)
- ten-frames
- walk-on number line

Multiplication and Division

When introducing multiplication, students should be introduced to multiplication through situations (equal-group story problems) that lend themselves to modelling with sets, arrays, and linear measurement models, such as number lines. For example, 3 x 4 can be represented with the

- Set Model – represent 3 groups of 4 birds (equal groups of birds) which equals 12 birds

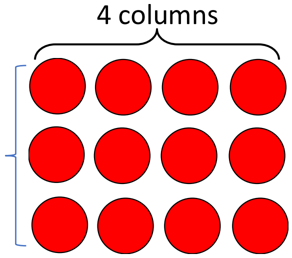
$4 + 4 + 4 = 12$ is connected to $3 \times 4 = 12$

This example shows a repeated addition of equal groups.

- Array Model – represent 3 rows of 4 groups of counters by making 3 rows of 4 columns

4 columns

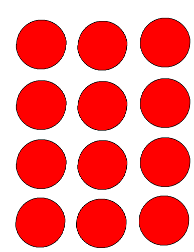
3 rows



$3 \times 4 = 12$

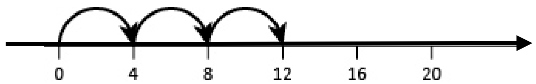
The first factor 3 represents the number of rows. The second factor 4 represents the number of columns.

Note: The array on the right, of 4 rows and 3 columns, represents the multiplication for 4 x 3.
The first array represents 3 x 4 = 12 counters.
The second array represents 4 x 3 = 12 counters.



Then, $3 \times 4 = 12$ and $4 \times 3 = 12$ which is the commutativity of the multiplication.
The students thinking is incorrect if they believe that the arrays are equal
4 x 3 = 12 because 3 x 4 = 4 x 3.

- Linear Model – represent 3 jumps of 4 on a number line



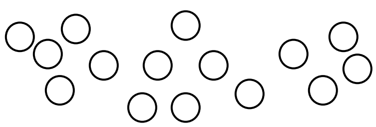
This representation is another way to show the multiplication 3 x 4 as a repeated addition (4 + 4 + 4). This can then be connected to the multiplication equation 3 x 4 = 12.

Division
Students should be introduced to division through story problems. For this instruction, there are two types of situations, equal-sharing and equal-grouping, which need to be considered. Equal-sharing problems are those in which the number of groups is known and the number in each group needs to be found. Equal-grouping problems are those in which the number in each group is known and the number of groups needs to be found. Students should also see that dividing the class into groups of 5, giving each student 4 pencils, and placing books into stacks of 4 are all examples of equal-grouping situations.

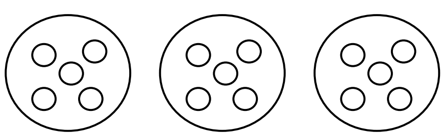
Students should see that dividing the class into two groups, sharing 12 pieces of paper with 4 students, and sharing a large bag of candy into 3 small bags are all examples of equal-sharing situations.

Below are some sample questions that represent division through story problems.

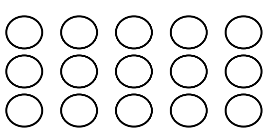
1. Three friends want to share 15 candies. How many candies will each friend get?



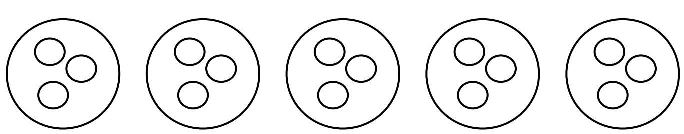
Each friend will get 5 candies. This may be described that when 15 is divided into 3 groups, there are 5 in each group.



2. Friends want to share 15 candies by each taking 3 candies. How many friends will get candies?



Five friends will get candies. This may be described that when 15 is divided into groups of 3, there are 5 groups.



Students should solve many examples of both types of division problems by modelling them concretely, recording them pictorially, and describing the division in words before they are introduced to division sentences.

D. What are the most appropriate methods and activities for assessing student learning?

Below are some sample questions related to operations with whole numbers which will be used to represent some of the appropriate methods and activities for assessing student learning.

Encourage students to share their thinking, their strategies, and their solutions. Rich discussion fosters the development of alternative problem-solving strategies.

Examples:

1. Choose the correct answer for the following addition 363 + 25

○ 308
○ 388
○ 618
○ 5113

2. Chose the correct answer for the following subtraction

$$\begin{array}{r} 809 \\ \underline{-\ 489} \end{array}$$

○ 320
○ 420
○ 480
○ 1298

3. Forty-two students were in the gym. Twenty-six of them were girls. How many were boys?

Choose the equation that shows how to work out this problem.

<table>
<tr><td>26</td><td>△</td></tr>
<tr><td colspan="2">42</td></tr>
</table>

○ 42 = 26 + △
○ 26 + 42 = △
○ △ − 26 = 42
○ 26 − △ = 42

4. Which number is missing in this equation? $\boxed{}$ = 7 + 9

○ 19
○ 16
○ 13
○ 6

5. Which number is missing in this equation $3 + \square = 12 - 6$

- ○ 1
- ○ 3
- ○ 6
- ○ 9

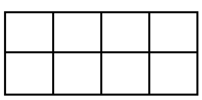
6. Luke has 72 stickers. Lilly has 29 stickers.
About how many more stickers does Luke have than Lilly?
Choose the best estimate.

- ○ 100
- ○ 90
- ○ 40
- ○ 30

7. Marley did this subtraction.
675 – 346 = 329
Which expression could help her check her work?

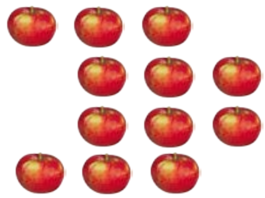
- ○ 675 + 329
- ○ 675 + 346
- ○ 329 + 346
- ○ 346 – 329

8. This array represents which multiplication fact?



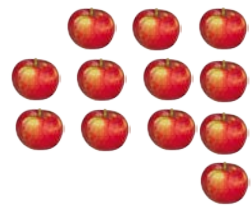
- ○ 1 x 8
- ○ 2 x 4
- ○ 2 x 8
- ○ 4 x 2

9. Look at the following pictures of apples.
Which picture represents the multiplication array 4×3?

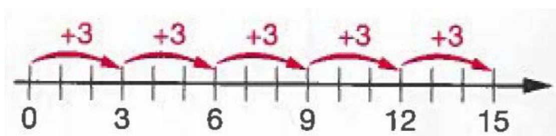
○ 

○ 

○ 

○ 

10. Look at the following number line:



Which operation is represented by this number line?

○ 3×5
○ 5×3
○ 15 + 3 + 3 + 3+ 3 + 3
○ 15×3

11. Which statement about the array below is **not true**?

○ The array has 2 rows and 4 columns.
○ The array represents the multiplication 4 x 2 = 8.
○ The array represents 4 + 4 = 8 items.
○ The array represents the multiplication 2 x 4 = 8.

12. Mila made 2 rectangular cakes. Each cake can be cut into 37 pieces.
About how many pieces of cake did Mila make?

- ○ 40
- ○ 70
- ○ 80
- ○ 90

13. Amir has 12 tomatoes. He puts the tomatoes into 3 boxes in equal groups.
Which equation below would help you determine the number of tomatoes in each box?

- ○ 12 + 3 = 15
- ○ 12 – 3 = 9
- ○ 12 x 3 = 36
- ○ 12 ÷ 3 = 4

14. An apple was cut into equal pieces.
Choose the fraction that would represent the smallest piece of apple.

- ○ $\frac{1}{4}$
- ○ $\frac{1}{2}$
- ○ $\frac{1}{8}$
- ○ $\frac{1}{10}$